Journal of Applied Mechanics and Technical Physics, Vol. 47, No. 3, pp. 439–445, 2006

OSCILLATIONS OF AN INHOMOGENEOUS ELASTIC LAYER

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UDC 539.3

A method is proposed to analyze wave fields in an elastic layer with elastic properties varying arbitrarily with depth. The method is based on reducing the boundary-value problem to a system of Fredholm integral equations of the second kind, which is then analyzed numerically. Some features of the structure of dispersion sets are analyzed and, in particular, their asymptotes are constructed.

Key words: *elastic inhomogeneous layer, steady-state oscillations, wave dispersion.*

Introduction. The problem of oscillations of inhomogeneous elastic waveguides has applications in various areas: geophysics, the mechanics of laminated composites, nano- and biomechanics. The features of wave fields in inhomogeneous layered structures were studied in [1–6]. In this case, the standard research procedure leads to a system of first-order differential equations, whose coefficients depend not only on the distribution of the Lamé coefficients on the cross-sectional coordinate but also on their derivatives [3, 4]. In this approach the important cases of the piecewise continuous inhomogeneity behavior remain unstudied. Kalinchuk and Belyankova [5] proposed a different approach based on numerical construction of linearly independent solutions of the system of linear differential equations with variable coefficient, which extends the class of functions studied.

The present paper deals with wave fields in an elastic layer with elastic properties varying arbitrarily with depth. The boundary-value problem reduces to a system of Fredholm integral equations of the second kind. Some features of the structure of dispersion sets are studied, and, in particular, their asymptotes are constructed.

1. Formulation of Boundary-Value Problems. We consider a layer ($|x_1|, |x_2| \leqslant \infty$, $0 \leqslant x_3 \leqslant h$) of nonuniform thickness with a rigidly fixed foundation ($x_3 = 0$) which performs steady-state oscillations at a frequency ω under a distributed load defined by the vector $\boldsymbol{p} = (p_1, p_2, p_3)\,\mathrm{e}^{-i\omega t}$. Plane and antiplane cases are studied. It is assumed that the Lamé parameters $\lambda = \lambda(x_3)$, $\mu = \mu(x_3)$ and the density of the layer $\rho(x_3)$ are arbitrary piecewise continuous functions. Problems of steady-state oscillations are considered. Below, the time coefficient $\mathrm{e}^{-i\omega t}$ is omitted.

Plane Deformation Problem. In the case of plane deformation, the components of the displacement vector u_1 and u_3 depend only on the coordinates x_1 and x_3 [$u_1 = u_1(x_1, x_3)$ and $u_3 = u_3(x_1, x_3)$] and $u_2 = 0$. The equations of motion are written as

$$\sigma_{11,1} + \sigma_{13,3} + \rho(x_3)\omega^2 u_1 = 0, \qquad \sigma_{31,1} + \sigma_{33,3} + \rho(x_3)\omega^2 u_3 = 0. \tag{1.1}$$

The Hooke's law is written as

$$\sigma_{11} = \lambda(x_3)(u_{1,1} + u_{3,3}) + 2\mu(x_3)u_{1,1}, \qquad \sigma_{33} = \lambda(x_3)(u_{1,1} + u_{3,3}) + 2\mu(x_3)u_{3,3},$$
$$\sigma_{13} = \mu(x_3)(u_{1,3} + u_{3,1}). \tag{1.2}$$

The boundary conditions correspond to rigid clamping of the lower boundary of the layer

$$u_1|_{x_3=0} = 0, \qquad u_3|_{x_3=0} = 0 \tag{1.3}$$

and loading on the upper boundary

$$\sigma_{13}|_{x_3=h} = p_1(x_1), \qquad \sigma_{33}|_{x_3=h} = p_3(x_1). \tag{1.4}$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 3, pp. 157–164, May–June, 2006. Original article submitted August 4, 2005.

Antiplane Deformation Problem. Among the displacement vector components, only the component $u_2 = u_2(x_1, x_3)$ is different from zero, and the equation of motion has the form

$$\sigma_{12,1} + \sigma_{23,3} + \rho(x_3)\omega^2 u_2 = 0, \tag{1.5}$$

where

$$\sigma_{12} = \mu(x_3)u_{2,1}, \qquad \sigma_{23} = \mu(x_3)u_{2,3}. \tag{1.6}$$

The boundary conditions correspond to rigid clamping of the lower boundary of the layer and loading on the upper boundary:

$$u_2|_{x_3=0} = 0; \tag{1.7}$$

$$\sigma_{23}|_{x_3=h} = p_2(x_1).$$

It should be noted that the formulated boundary-value problems lead to partial equations with variable coefficients.

2. Reducing the Boundary-Value Problems to Fredholm Integral Equations of the Second Kind. In the case of plane deformation, the boundary-value problem (1.1)–(1.4) is reduced to a system of first-order differential equations. This is done using the Fourier transform on the variable x_1:

$$\tilde{u}_1(\alpha, x_3) = \int_{-\infty}^{\infty} u_1(x_1, x_3)\,\mathrm{e}^{i\alpha x_1}\,dx_1, \qquad \tilde{u}_3(\alpha, x_3) = \int_{-\infty}^{\infty} u_3(x_1, x_3)\,\mathrm{e}^{i\alpha x_1}\,dx_1,$$

$$\tilde{p}_1(\alpha) = \int_{-\infty}^{\infty} p_1(x_1)\,\mathrm{e}^{i\alpha x_1}\,dx_1, \qquad \tilde{p}_3(\alpha) = \int_{-\infty}^{\infty} p_3(x_1)\,\mathrm{e}^{i\alpha x_1}\,dx_1.$$

The result is the canonical system of first-order ordinary differential equations for the transforms:

$$\begin{gathered} \frac{d\tilde{u}_1}{dx_3} = \alpha\tilde{u}_3 + \frac{1}{\mu}\,\tilde{\sigma}_{13}, \qquad \frac{d\tilde{u}_3}{dx_3} = \frac{i\alpha\lambda}{\lambda+2\mu}\,\tilde{u}_1 + \frac{1}{\lambda+2\mu}\,\tilde{\sigma}_{33}, \\ \frac{d\tilde{\sigma}_{13}}{dx_3} = \Big(\frac{4\alpha^2\mu(\lambda+\mu)}{\lambda+2\mu} - \rho\omega^2\Big)\tilde{u}_1 + \frac{i\alpha\lambda}{\lambda+2\mu}\,\tilde{\sigma}_{33}, \qquad \frac{d\tilde{\sigma}_{33}}{dx_3} = -\rho\omega^2\tilde{u}_3 + i\alpha\tilde{\sigma}_{31}. \end{gathered} \tag{2.1}$$

The parameters are rendered dimensionless by introducing the following notation:

$$z = \frac{x_3}{h} \quad (x_3 \in [0,h] \mapsto z \in [0,1]), \quad \hat{\mu}(z) = \frac{\mu(hz)}{\mu_0}, \quad \hat{\lambda}(z) = \frac{\lambda(hz)}{\mu_0}, \quad \hat{p}_1 = \frac{\tilde{p}_1}{\mu_0}, \quad \hat{p}_3 = \frac{\tilde{p}_3}{\mu_0},$$

$$W_{13}(z) = \frac{\tilde{\sigma}_{13}(hz)}{\mu_0}, \quad W_{33}(z) = \frac{\tilde{\sigma}_{33}(hz)}{\mu_0}, \quad V_1(z) = \frac{\tilde{u}_1(hz)}{h}, \quad V_3(z) = \frac{\tilde{u}_3(hz)}{h},$$

$$\hat{\rho}(z) = \frac{\rho(hz)}{\rho_0}, \qquad \beta = \alpha h, \qquad \varkappa^2 = \frac{\rho_0\omega^2h^2}{\mu_0}.$$

Here ρ_0 and μ_0 are the characteristic density and shear modulus. Then, system (2.1) becomes

$$\begin{gathered} V_1' = i\beta V_3 + \frac{1}{\hat{\mu}}\,W_{13}, \qquad V_3' = \frac{i\beta\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}\,V_1 + \frac{1}{\hat{\lambda}+2\hat{\mu}}\,W_{33}, \\ W_{13}' = \Big(\frac{4\beta^2\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}} - \varkappa^2\hat{\rho}\Big)V_1 + \frac{i\beta\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}\,W_{33}, \qquad W_{33}' = -\varkappa^2\hat{\rho}V_3 + i\beta W_{31}, \end{gathered} \tag{2.2}$$

and boundary conditions (1.3) and (1.4) become

$$\begin{gathered} V_1|_{z=0} = 0, \qquad V_3|_{z=0} = 0, \\ W_{13}|_{z=1} = \hat{p}_1(\alpha), \qquad W_{33}|_{z=1} = \hat{p}_3(\alpha). \end{gathered} \tag{2.3}$$

Integrating the system of differential equations (2.2) on the segment $[0,z]$ and finding the integration constants from boundary conditions (2.3), we obtain

$$V_1 = \int_0^z \Big(i\beta V_3 + \frac{1}{\hat{\mu}} W_{13} \Big) \, d\xi, \qquad V_3 = \int_0^z \Big(\frac{i\beta\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} V_1 + \frac{1}{\hat{\lambda}+2\hat{\mu}} W_{33} \Big) \, d\xi,$$

$$W_{13} = -\int_z^1 \Big[\Big(\frac{4\beta^2\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}} - \varkappa^2\hat{\rho} \Big) V_1 + \frac{i\beta\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} W_{33} \Big] \, d\xi + \hat{p}_1,$$

$$W_{33} = \int_z^1 \Big(\varkappa^2 \hat{\rho} V_3 - i\beta W_{31} \Big) \, d\xi + \hat{p}_3.$$

Eliminating V_3 and W_{13} from this system, changing the order of integration in the double integrals, and introducing the notation $U = V_1$ and $T = iW_{33}$, we obtain the following system of Fredholm integral equations of the second kind:

$$\begin{gathered} U(z) = \int_0^1 M_1(z,\xi) U(\xi) \, d\xi + \int_0^1 M_2(z,\xi) T(\xi) \, d\xi + \hat{p}_1 \int_0^z \frac{1}{\hat{\mu}(\xi)} \, d\xi, \\ -T(z) = \int_0^1 M_3(z,\xi) T(\xi) \, d\xi + \int_0^1 M_4(z,\xi) U(\xi) \, d\xi - i\hat{p}_3 - \beta \hat{p}_1 (1-z). \end{gathered} \tag{2.4}$$

The kernels of the integral operators can be written as

$$M_1(z,\xi) = -\frac{\beta^2\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} K_1(z,\xi) - \Big(\frac{4\beta^2\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}} - \varkappa^2\hat{\rho} \Big) K_2(z,\xi),$$

$$M_2(z,\xi) = \frac{\beta}{\hat{\lambda}+2\hat{\mu}} K_1(z,\xi) - \frac{\beta\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} K_2(z,\xi),$$

$$M_3(z,\xi) = -\frac{\varkappa^2\hat{\rho}}{\hat{\lambda}+2\hat{\mu}} K_3(z,\xi) + \frac{\beta^2\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} K_4(z,\xi),$$

$$M_4(z,\xi) = \beta \Big[\frac{\varkappa^2\hat{\rho}\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} K_3(z,\xi) + \Big(\frac{4\beta^2\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}} - \varkappa^2\hat{\rho} \Big) K_4(z,\xi) \Big],$$

$$K_1(z,\xi) = (z-\xi)\theta(z-\xi), \qquad K_2(z,\xi) = \int_0^{\min\{z;\xi\}} \frac{1}{\hat{\mu}(\tau)} \, d\tau,$$

$$K_3(z,\xi) = \min\{1-z; 1-\xi\}, \qquad K_4(z,\xi) = K_1(\xi,z),$$

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Similarly, the boundary-value problem of antiplane deformation (1.5)–(1.7) is reduced to the system of first-order differential equations

$$W'(z) = -F(z)V(z), \qquad V'(z) = W(z)/\hat{\mu}(z) \tag{2.5}$$

with the boundary conditions

$$V|_{z=0} = 0, \qquad W|_{z=1} = \hat{p}_2(\alpha), \tag{2.6}$$

where

$$\hat{p}_2 = \frac{\tilde{p}_2}{\mu_0}, \quad W(z) = \frac{\tilde{\sigma}_{23}(z)}{\mu_0}, \quad V(z) = \frac{\tilde{u}_2(z)}{h}, \quad F(z) = \varkappa^2 \hat{\rho}(z) - \beta^2 \hat{\mu}(z).$$

Similarly to the plane deformation problem, problem (2.5), (2.6) is reduced to the Fredholm integral equation of the second kind:

$$V(z) = \int_0^1 K(z,\xi) V(\xi)\, d\xi + f(z), \tag{2.7}$$

where

$$K(z,\xi) = F(\xi) \int_0^{\min\{z;\xi\}} \frac{d\tau}{\hat{\mu}(\tau)}, \qquad f(z) = \hat{p}_2 \int_0^z \frac{d\xi}{\hat{\mu}(\xi)}.$$

It should be noted that the constructed integral equations can be solved numerically if the corresponding inhomogeneity behavior is specified. For some combinations of parameters, the corresponding equations can be unsolvable, which characterizes the points of dispersion sets [1].

To construct dispersion sets, it is necessary to analyze the Fredholm homogeneous integral equations of the second kind (2.4) and (2.7) and to find combinations of the parameters $\varkappa$ and β for which the corresponding integral equations have nontrivial solutions [3].

3. Numerical Analysis of Fields and Dispersion Sets. In this study, the Fredholm integral equation of the second kind is discretized by dividing the segment $[0, 1]$ into $n - 1$ segments by the points $z_1 = 0 < z_2 < \ldots < z_n = 1$, $\Delta z_i = z_{i+1} - z_i$. On each of these segments, the unknown function $V(z)$ is approximated by the linear function $\tilde{V}(z)$:

$$\tilde{V}(z) = V_{i+1} \frac{z - z_i}{\Delta z_i} - V_i \frac{z - z_{i+1}}{\Delta z_i} \qquad \text{at} \quad z \in [z_i, z_{i+1}], \quad i = 1, \ldots, n-1.$$

Then, satisfying the integral equation (2.7) at the points z_i, we obtain the following linear algebraic system for the nodal unknowns $V_i = V(z_i)$:

$$V_i + \sum_{j=1}^{n} C_{ji} V_j = f(z_i), \qquad i = 1, \ldots, n,$$

$$C_{1i} = \int_{z_1}^{z_2} K(z_i,\xi) \frac{\xi - z_2}{\Delta z_1}\, d\xi, \qquad C_{ni} = -\int_{z_{n-1}}^{z_n} K(z_i,\xi) \frac{\xi - z_{n-1}}{\Delta z_{n-1}}\, d\xi, \qquad i = 1, \ldots, n,$$

$$C_{ji} = \int_{z_j}^{z_{j+1}} K(z_i,\xi) \frac{\xi - z_{j+1}}{\Delta z_j}\, d\xi - \int_{z_{j-1}}^{z_j} K(z_i,\xi) \frac{\xi - z_{j-1}}{\Delta z_{j-1}}\, d\xi, \qquad j = 2, \ldots, n-1.$$

According to the adopted discretization procedure, the problems considered are reduced to systems of linear equations, which are unsolvable for some combinations of the parameters $\varkappa$ and β. The set of such points $(\varkappa, \beta)$ form the dispersion set in the primal problems. To find the points of a dispersion set, it is necessary to find combinations of the parameters $\varkappa$ and β such that the corresponding homogeneous system of linear equations has a nontrivial solution. Some general regularities in the structure of these sets are given in [1, 2]. The real component of a dispersion set consists of a finite number of smooth curves originating at some points on the $\beta = 0$ axis that are defined as the eigenvalues of the following independent spectral problems:

$$(\hat{\mu} V_1')' + \varkappa^2 \hat{\rho} V_1 = 0, \qquad V_1(0) = V_1'(0) = 0,$$

$$W_{33}' + \frac{\varkappa^2 \hat{\rho}}{\hat{\lambda} + 2\hat{\mu}} W_{33} = 0, \qquad W_{33}(1) = W_{33}'(0) = 0.$$

These problems result in the eigenvalues $\varkappa_k^2$ $(k = 1, 2, \ldots, N)$ of positive self-conjugate operators that in the case of a homogeneous medium correspond to the barrier frequencies of transverse and longitudinal waves and generate

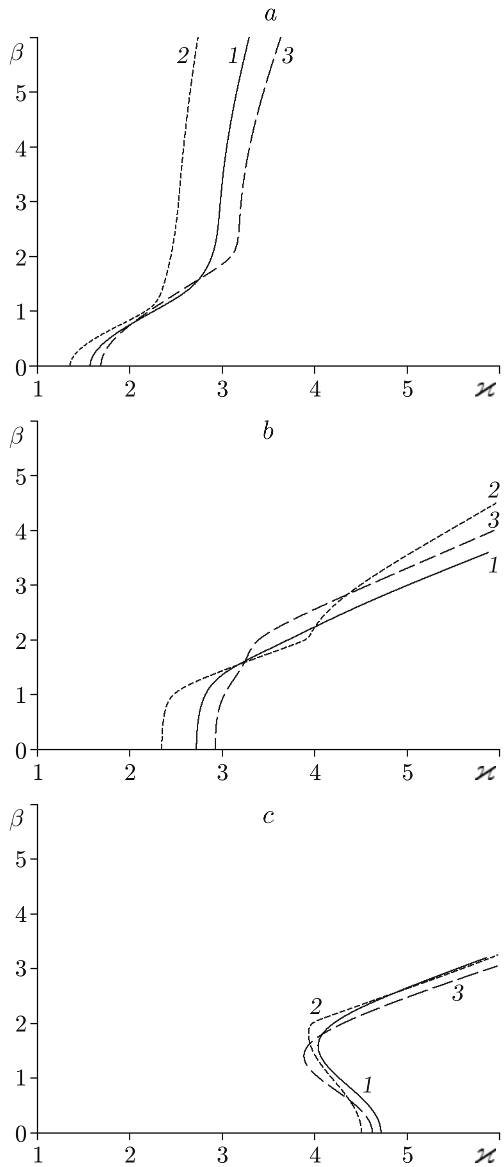

Fig. 1

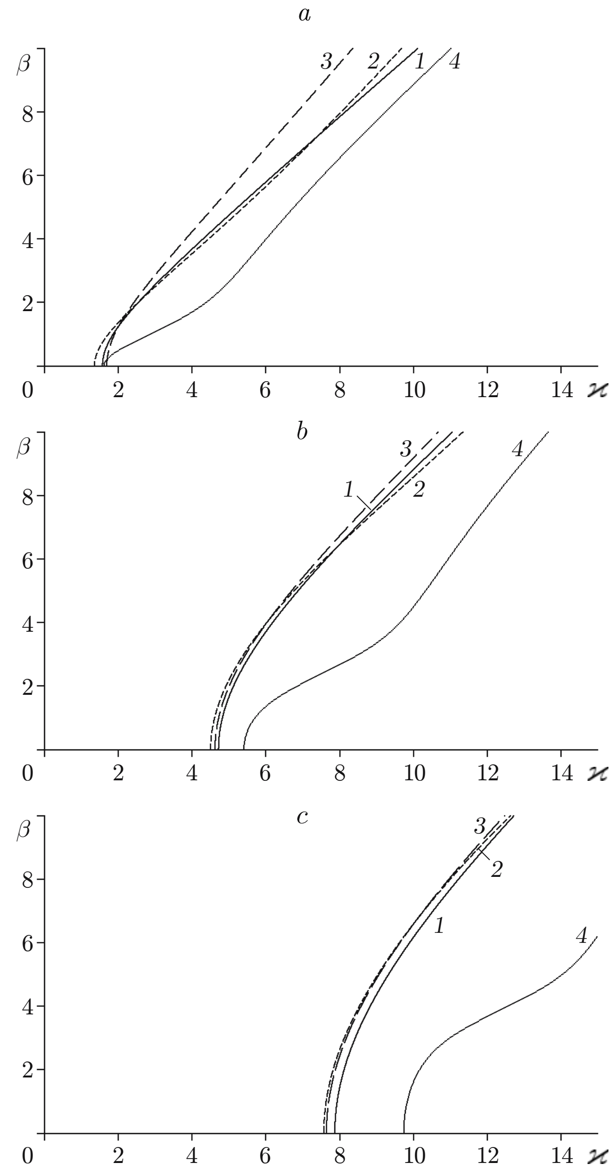

Fig. 2

Fig. 1. Branches of the dispersion set of the plane deformation problem: (a) first branch; (b) second branch; (c) third branch; 1) $\hat{\lambda} = 1$, $\hat{\mu} = 1$, and $\hat{\rho} = 1$; 2) $\hat{\lambda} = z + 1/2$, $\hat{\mu} = z + 1/2$, and $\hat{\rho} = 1$; 3) $\hat{\lambda} = 3/2 - z$, $\hat{\mu} = 3/2 - z$, and $\hat{\rho} = 1$.

Fig. 2. Branches of the dispersion set for the antiplane deformation problem: curve 4 refers to $\hat{\rho} = 1$ and $\hat{\mu} = 1$ if $z \in [0, 2/3]$ or 10 if $z \in (2/3, 1]$; the remaining notation same as in Fig. 1.

the corresponding branches of the dispersion set. In this case, there exists the critical wavenumber $\varkappa_*$ such that in the region $\varkappa < \varkappa_*$ there are no points of the dispersion set [2].

Dispersion sets for the different inhomogeneity behaviors are constructed numerically for $n = 40$. Figure 1 gives the first three branches of the dispersion set for the plane deformation problem for some parameters of the medium. The inhomogeneity behavior is specified so that the integrals of all examined functions are equal to unity.

It should be noted that abnormal dispersion is observed on the third branch, and it is the most pronounced for the third inhomogeneity behavior.

Results of similar calculations for the plane deformation problem are given in Fig. 2.

We also note that for $\mu = \mu_0 = \text{const}$ and $\rho = \rho_0 = \text{const}$, the dispersion set can be constructed analytically and is a set of hyperbolas:

$$\varkappa^2 - \beta^2 = \pi^2(1/2 + n)^2, \qquad n = 0, 1, 2, \ldots, \qquad \varkappa_* = \pi/2.$$

4. Structure of the Dispersion Sets. We study some common regularities in the structure of the dispersion sets.

We consider system (2.5) with homogeneous boundary conditions in the antiplane deformation problem. Eliminating $W(z)$ from this system, we write the following equation for $V(z)$:

$$(\hat{\mu}(z)V'(z))' = -F(z)V(z), \qquad V'(1) = 0.$$

Multiplying it by $V(z)$, integrating on the segment $[0, 1]$, and performing some transformations, we obtain the main identity for the dispersion curves:

$$\varkappa^2 = \beta^2 \int\limits_0^1 \hat{\mu}(V^2 + V'^2)\, dz \Big/ \int\limits_0^1 \hat{\rho} V^2\, dz. \tag{4.1}$$

Using the notation $\mu_{\max} = \max\limits_{z\in[0,1]} \hat{\mu}(z)$, $\mu_{\min} = \min\limits_{z\in[0,1]} \hat{\mu}(z)$, $\rho_{\max} = \max\limits_{z\in[0,1]} \hat{\rho}(z)$, and $\rho_{\min} = \min\limits_{z\in[0,1]} \hat{\rho}(z)$, from the main identity (4.1), we obtain the following estimate for the wavenumber:

$$\beta^2\mu_{\min}/\rho_{\max} + C^- \leqslant \varkappa^2 \leqslant \beta^2\mu_{\max}/\rho_{\min} + C^+.$$

Here the quantities C^- and C^+ satisfy the inequalities

$$C^- \leqslant \int\limits_0^1 \hat{\mu} V'^2\, dz \Big/ \int\limits_0^1 \hat{\rho} V^2\, dz \leqslant C^+.$$

The quantity C^- is easy to estimate using the Cauchy–Bunyakovskii inequality

$$C^- = 1 \Big/ \rho_{\max} \int\limits_0^1 \Big(\int\limits_0^t \frac{d\xi}{\hat{\mu}(\xi)} \Big)\, dt.$$

The asymptotics $V(1)$ as $|\beta| \to \infty$ is constructed as follows. System (2.5) is written as

$$\begin{gathered} W'(z) = \beta^2\hat{\mu}(z)V(z), \qquad W(z) = \hat{\mu}(z)V'(z), \\ V(0) = 0, \qquad W(1) = \hat{p}_2. \end{gathered} \tag{4.2}$$

The solution of system (4.2) is sought in the form $V(z) = A\, \mathrm{e}^{\beta S(z)}$ $(\beta > 0)$. Substituting this representation into the starting equation and retaining terms of higher order in β, we obtain $S(z) = \pm z$. Then, $V(z) = A_1\, \mathrm{e}^{\beta z} + A_2\, \mathrm{e}^{-\beta z}$. Satisfying the boundary conditions, we have

$$V(1) = \frac{\hat{p}_2 \sinh \beta}{\beta\mu(1)\cosh \beta} \sim \frac{\hat{p}_2}{\beta\mu(1)}.$$

Let us construct the asymptotics for the dispersion curves for $|\beta| \to \infty$. Setting $\varkappa = t\beta$, we write system (2.5) with homogeneous boundary conditions in the form

$$\begin{gathered} W'(z) = \beta^2(\hat{\mu}(z) - \hat{\rho}(z)t^2)V(z), \qquad W(z) = \hat{\mu}(z)V'(z), \\ V(0) = 0, \qquad W(1) = 0. \end{gathered}$$

Its solution is sought in the form $V(z) = A\, \mathrm{e}^{\beta S(z)}$. Then,

$$S(z) = \pm \int\limits_0^z \sqrt{1 - \frac{\hat{\rho}(\xi)t^2}{\hat{\mu}(\xi)}}\, d\xi.$$

Satisfying the boundary conditions, we obtain $\beta \cosh\,(\beta S(1))S'(1) = 0$, whence $t = \sqrt{\hat{\mu}(1)/\hat{\rho}(1)}$. Thus, the asymptote of the curves of the dispersion set is determined by the velocity of the transverse waves of an elastic medium with characteristics on the upper boundary.

Numerical calculations for constructing the points of the dispersion set show that for $\beta > 10$, the calculation results and the results of the asymptotic analysis differ by not more than 2% for all kinds of inhomogeneity.

Let us construct the asymptotics of the points of the dispersion sets for $|\beta| \to \infty$ in the plane deformation problem. Setting $\varkappa = t\beta$, we obtain a solution of system (2.2): $V_1 = A_1\,\mathrm{e}^{\beta S(z)}$ and $V_3 = A_3\,\mathrm{e}^{\beta S(z)}$. From the first two equations (2.2), W_{13} and W_{33} are expressed as

$$W_{13} = \hat{\mu}[A_1 S'(z) - iA_3]\beta\,\mathrm{e}^{\beta S(z)}, \qquad W_{33} = (\hat{\lambda} + 2\hat{\mu})\Big(A_3 S'(z) - i\,\frac{\hat{\lambda}}{\hat{\lambda} + 2\hat{\mu}}\,A_1\Big)\beta\,\mathrm{e}^{\beta S(z)}\,.$$

Retaining only the main terms of the asymptotic expansions in the last two equations (2.2), we formulate a system for the constants A_1 and A_3. Setting the determinant of this system equal to zero, we have

$$S_1 = \pm\int\limits_0^z \sqrt{1 - \frac{\hat{\rho}(\xi)t^2}{\hat{\lambda}(\xi) + 2\hat{\mu}(\xi)}}\,d\xi, \qquad S_2 = \pm\int\limits_0^z \sqrt{1 - \frac{\hat{\rho}(\xi)t^2}{\hat{\mu}(\xi)}}\,d\xi.$$

Satisfying the homogeneous boundary conditions on the boundaries of the layer, we obtain the corresponding equation of the form

$$[1 - S_2'(0)S_1'(0)]\,\mathrm{e}^{\beta(S_1(1)+S_2(1))}\{4S_1'(1)S_2'(1) - [1 + S_2'^2(1)]\} = 0.$$

It should be noted that the vanishing of the first term gives a physically meaningless value for t, and the vanishing of the third term leads to the well-known Rayleigh equation [5] with elastic constants on the upper boundary of the layer.

This work was supported by the Russian Foundation for Basic Research (Grant No. 05-01-00734) and the Foundation "Leading Scientific Schools of Russia" (Grant No. NSh.-5014.2006.1).

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